Exact and approximate expressions for the period of anharmonic oscillators

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In this paper we present a straightforward systematic method for the exact and approximate calculation of integrals that appear in formulas for the period of anharmonic oscillators and other problems of interest in classical mechanics.

PACS numbers: 45.10.Db, 04.25.-g

I. INTRODUCTION

The discussion of periodic motion in one dimension is important in most introductory courses on classical mechanics. Several problems can be solved exactly, but in most cases one has to resort to approximate solutions. Simple but sufficiently accurate approximate solutions for such problems are very important in understanding many features of classical mechanics. In addition to it, in some cases one is simply satisfied with accurate numerical results, and expressions suitable for computation are most welcome.

The purpose of this paper is the discussion of the exact and approximate calculation of the period of a particle that moves in one dimension under the effect of an anharmonic potential.

II. PERIODIC MOTION IN ONE DIMENSION

Consider a particle of mass m moving in one dimension under a potential–energy function $V(x)$. Without loss of generality we assume that $V(x)$ has a minimum at $x=0$; more precisely, we assume that $V(0)=0$, $V'(0)=0$, and $V''(0)>0$, where the prime indicates differentiation with respect to x. Following the standard notation in classical mechanics, we use a dot to indicate differentiation with respect to time, for example: $v=\dot{x}$.

¿From the equation of motion

$$m\ddot{x} = -V'(x) \tag{1}$$

we easily obtain an integral of the motion

$$E = \frac{m\dot{x}^2}{2} + V(x) \tag{2}$$

which is the total energy. The motion of the particle is restricted to the interval $x_- < x < x_+$, where the turning points $x_\pm$ satisfy $V(x_\pm) = E$; that is to say, $\dot{x} = 0$ at those points.

It is well–known that the period of the motion is given by

$$T = \oint dt = \sqrt{2m} \int_{x_-}^{x_+} \frac{dx}{\sqrt{E - V(x)}} \tag{3}$$

from which we obtain the frequency $\Omega = 2\pi/T$.

We can simplify the equations of motion by the introduction of a dimensionless time $\tau = \omega_0 t$, where ω_0 is an arbitrary frequency. If we define

$$\mathcal{E} = \frac{E}{m\omega_0^2},\ U(x) = \frac{V(x)}{m\omega_0^2} \tag{4}$$

then we obtain the equations of motion of a particle of unit mass; for example:

$$\mathcal{E} = \frac{\dot{x}^2}{2} + U(x) \tag{5}$$

and

$$T = \frac{\sqrt{2}}{\omega_0} \int_{x_-}^{x_+} \frac{dx}{\sqrt{\mathcal{E} - U(x)}}. \tag{6}$$

It is worth noticing that equation (5) is not dimensionless because $\mathcal{E}$ and $U(x)$ have units of length squared. In order to get a truly dimensionless equation we should define a dimensionless coordinate $q = x/L$, where L has units of length. Thus $E/\left(m\omega_0^2 L^2\right)$ and $V(Lq)/\left(m\omega_0^2 L^2\right)$ are the dimensionless counterparts of the total and potential energies, respectively. In this paper, however, we have opted for equations that are similar to those often found in current literature.

For example, if

$$V(x) = \frac{v_2 x^2}{2} + \frac{v_4 x^4}{4} \tag{7}$$

we may choose $\omega_0 = \sqrt{v_2/m}$ so that

$$U(x) = \frac{x^2}{2} + \frac{\lambda x^4}{4}, \tag{8}$$

where $\lambda = v_4/v_2$.

III. THE MAIN INTEGRAL

It follows from the discussion above that the period is proportional to an integral of the form

$$I = \int_{x_-}^{x_+} \frac{dx}{\sqrt{Q(x)}}, \tag{9}$$

were $Q(x)$ exhibits simple zeros at x_- and x_+ and is positive definite for all $x_- < x < x_+$. That is to say, we can write

$$Q(x) = (x_+ - x)(x - x_-)R(x) \tag{10}$$

where $R(x) > 0$ for all $x_- \leq x \leq x_+$.

The reason for rewriting our problem in this somewhat abstract way is that the integral (9) applies to problems other than the period of a motion in one dimension. We will mention some examples later on.

In order to develop suitable exact and approximate expressions for the integral (9) we define the reference function

$$Q_0(x) = \frac{\omega^2}{2}\left(x_+ - x\right)\left(x - x_-\right) \tag{11}$$

that satisfies the appropriate boundary conditions at the turning points. It is clear that $Q_0(x)$ is the function that would appear in the treatment of a harmonic oscillator. Then we rewrite (9) as

$$I = \int_{x_-}^{x_+} \frac{dx}{\sqrt{Q_0(x)}\sqrt{1+\Delta(x)}}\ , \tag{12}$$

where

$$\Delta(x) \equiv \frac{Q(x) - Q_0(x)}{Q_0(x)} = \frac{2R(x) - \omega^2}{\omega^2}. \tag{13}$$

The change of variables

$$x = \frac{x_+ + x_-}{2} + \frac{x_+ - x_-}{2}\ \cos\theta \tag{14}$$

makes the integral (12) much simpler:

$$I = \frac{\sqrt{2}}{\omega}\int_0^\pi \frac{d\theta}{\sqrt{1+\Delta}}. \tag{15}$$

This equation leads to an exact expression for the period, which in most cases one has to calculate numerically. In order to derive simple analytical formulas we expand

$$\frac{1}{\sqrt{1+\Delta}} = \sum_{j=0}^{\infty}\begin{pmatrix} -1/2 \\ j \end{pmatrix}\Delta^j \tag{16}$$

where $\begin{pmatrix} a \\ b \end{pmatrix} = a!/[b!(a-b)!]$ is a combinatorial number. Notice that this series converges for all x such that $|\Delta| < 1$. We thus obtain a series for the integral (15):

$$I = \sum_{j=0}^{\infty} I_j,\ I_j = \frac{\sqrt{2}}{\omega}\begin{pmatrix} -1/2 \\ j \end{pmatrix}\int_0^\pi \Delta^j\, d\theta. \tag{17}$$

In this way we can derive approximate expressions for the integral (15) by means of the partial sums:

$$I^{(N)} = \sum_{j=0}^{N} I_j. \tag{18}$$

IV. THE DUFFING OSCILLATOR

The potential–energy function (8) gives rise to the Duffing oscillator. Since it is parity invariant ($U(-x) = U(x)$) then $x_+ = -x_- = A$ is the amplitude of the oscillations. According to the general discussion of the preceding section, it follows from

$$Q(x) = \mathcal{E} - U(x) = \left(A^2 - x^2\right)\left[\frac{1}{2} + \frac{\lambda}{4}\left(A^2 + x^2\right)\right] \tag{19}$$

that

$$R(x) = \frac{1}{2} + \frac{\lambda}{4}\left(A^2 + x^2\right) \tag{20}$$

and

$$\Delta = \frac{1 + \lambda A^2 - \omega^2 - \frac{\lambda A^2}{2}\sin^2\theta}{\omega^2} \tag{21}$$

where we have substituted $x = A\cos\theta$. We conclude that the period depends on the dimensionless parameter $\rho = \lambda A^2$ that is the ratio of $v_4 A^4$ and $v_2 A^2$ both having units of energy.

If we choose $\omega = \sqrt{1+\rho}$ then we obtain an already known suitable compact expression for the integral[1]

$$I = \frac{\sqrt{2}}{\sqrt{1+\rho}}\int_0^\pi \frac{d\,\theta}{\sqrt{1-\xi\sin^2\theta}},\ \xi = \frac{\rho}{2\rho+2}. \tag{22}$$

This equation yields the series

$$I = \frac{\sqrt{2}\pi}{\sqrt{1+\rho}}\sum_{j=0}^{\infty}\begin{pmatrix} -1/2 \\ j \end{pmatrix}^2 \xi^j \tag{23}$$

that converges for all $|\xi| < 1$; that is to say, for all $\rho > -2/3$ or $\rho < -2$.

When $\lambda < 0$ the potential exhibits two barriers of height $1/(-4\lambda)$ at $x = \pm 1/\sqrt{-\lambda}$ and therefore the amplitude of the periodic motion cannot be greater than $A_L = 1/\sqrt{-\lambda}$. In other words, there is periodic motion if $\rho > \rho_L = \lambda A_L^2 = -1$. The series (23) does not converge for $-1 < \rho < -2/3$ and the analytical expressions that we may derive from it will not be valid for all the values of the energy that give rise to periodic motion. Can we improve this approach?. The answer is "yes " as we will see below.

Let R_M and R_m be the maximum and minimum values of $R(x)$ in the interval $[x_-, x_+]$ and Δ_M and Δ_m the corresponding values of $\Delta(x)$. Since $R(x)$ is positive definite we know that $R_M \geq R(x) \geq R_m > 0$. If we choose the value of the adjustable parameter ω so that $\Delta_M = -\Delta_m$ we obtain

$$\omega_b^2 = R_M + R_m > 0 \tag{24}$$

and

$$\Delta_b(x) = \frac{2R(x) - R_M - R_m}{R_M + R_m}. \tag{25}$$

The subscript b indicates that this particular value of ω "balances" the maximum and minimum values of $\Delta(x)$. Notice that $|\Delta(x)| < 1$ for all $x_- \leq x \leq x_+$ because $\Delta_M = (R_M - R_m)/(R_M + R_m) < 1$.

For the particular case of the Duffing oscillator we have $R_m = R(0) = 1/2 + \rho/4$ and $R_M = R(\pm A) = 1/2 + \rho/2$ so that

$$\omega_b^2 = \frac{4 + 3\rho}{4} \tag{26}$$

and

$$\Delta_b(\theta) = \frac{\rho}{4 + 3\rho}\cos(2\theta). \tag{27}$$

Thus the integral becomes

$$I = \frac{2\sqrt{2}}{\sqrt{4+3\rho}} \int_0^{\pi} \frac{d\theta}{\sqrt{1+\xi\cos(2\theta)}},\ \xi = \frac{\rho}{4+3\rho} \tag{28}$$

that gives rise to the series

$$I = \frac{2\sqrt{2}\pi}{\sqrt{4+3\rho}} \sum_{j=0}^{\infty} (-1)^j \begin{pmatrix} -1/2 \\ j \end{pmatrix} \begin{pmatrix} -1/2 \\ 2j \end{pmatrix} \xi^{2j}. \tag{29}$$

which converges for all $|\xi| < 1$; that is to say, for all $\rho > -1$ or $\rho < -2$. In this way we may obtain simple analytical expressions for the period valid for all values of the energy consistent with periodic motion.

According to equation (6) the period is given by

$$T = \frac{\sqrt{2}}{\omega_0} I \tag{30}$$

and equation (29) enables us to derive simple analytical approximate expressions for it. For concreteness and simplicity we choose $\omega_0 = 1$ in what follows. For example, the first two approximations are

$$T^{(0)} = \frac{4\pi}{\sqrt{4+3\rho}} \tag{31}$$

and

$$T^{(1)} = \frac{\pi\left(147\rho^2 + 384\rho + 256\right)}{4\left(4+3\rho\right)^{5/2}}. \tag{32}$$

These expressions are expected to be accurate for small values of ρ, and in fact they give the exact result for $\rho = 0$. However, they are also accurate for extremely great values of ρ. Notice that

$$\lim_{\rho\to\infty} \sqrt{\rho} T = 4 \int_0^{\pi} \frac{d\theta}{\sqrt{3+\cos(2\theta)}} \approx 7.4162987 \tag{33}$$

A straightforward calculation shows that

$$\lim_{\rho\to\infty} \sqrt{\rho} T^{(0)} = \frac{4\pi}{\sqrt{3}} \approx 7.26 \tag{34a}$$

$$\lim_{\rho\to\infty} \sqrt{\rho} T^{(1)} = \frac{49\sqrt{3}\pi}{36} \approx 7.406. \tag{34b}$$

We conclude that such simple analytical expressions are sufficiently accurate for most purposes and that one easily improves them by straightforward addition of more terms of the series (29).

V. QUADRATIC–CUBIC OSCILLATOR

Parity–invariant oscillators exhibit symmetric turning points; if the potential is nonsymmetric so are the turning points. The simplest example is

$$V(x) = \frac{v_2}{2} x^2 + \frac{v_3}{3} x^3. \tag{35}$$

If we again choose $\omega_0 = \sqrt{v_2/m}$ then we obtain

$$U(x) = \frac{x^2}{2} + \frac{\lambda}{3} x^3,\ \lambda = \frac{v_3}{v_2}. \tag{36}$$

The potential–energy function $U(x)$ shows a barrier of height $U(x_M) = 1/\left(6\lambda^2\right)$ at $x_M = -1/\lambda$, and the turning points satisfy $x_+ > 0 > x_-$.

If we write

$$Q(x) = \mathcal{E} - U(x) = (x - x_-)(x_+ - x)(b_0 + b_1 x) \tag{37}$$

then we obtain

$$b_0 = -\frac{x_+ x_-}{2(x_+^2 + x_+ x_- + x_-^2)} \quad , \quad b_1 = \frac{\lambda}{3} = -\frac{x_+ + x_-}{2(x_+^2 + x_+ x_- + x_-^2)} \tag{38}$$

and

$$\lambda = -\frac{3}{2}\,\frac{x_- + x_+}{x_+^2 + x_+ x_- + x_+^2}. \tag{39}$$

Since $U(-x, -\lambda) = U(x, \lambda)$ we consider only the case $\lambda > 0$ without loss of generality; therefore $x_- + x_+ < 0$ because $x_+^2 + x_+ x_- + x_-^2 > 0$. Taking into account that $b_0 > 0$ and $b_1 > 0$ we conclude that R_m and R_M take place at the turning points; therefore,

$$\omega_b^2 = R(x_+) + R(x_-) = -\frac{x_+^2 + 4x_+ x_- + x_-^2}{2\left(x_+^2 + x_+ x_- + x_-^2\right)}. \tag{40}$$

Is ω_b real for all values of $\mathcal{E}$ below the barrier?. In order to answer this question notice that the third root x_3 of $Q(x)$ is smaller than x_- and is given by

$$x_3 = -\frac{b_0}{b_1} = -\frac{x_+ x_-}{x_+ + x_-} < 0. \tag{41}$$

Therefore

$$\begin{aligned} & x_+^2 + 4x_+ x_- + x_-^2 \\ & \qquad = (x_+ + x_-)^2 + 2x_+ x_- \\ & \qquad = (x_+ + x_-)\,(x_+ + x_- - 2x_3) < 0 \end{aligned} \tag{42}$$

because $x_- - x_3 > 0$ and $x_+ - x_3 > 0$.

Finally, after the change of variables (14) the function $\Delta(\theta)$ takes a particularly simple form:

$$\Delta_b(\theta) = \xi \cos\theta,\ \xi = \frac{(x_+^2 - x_-^2)}{x_+^2 + 4x_+ x_- + x_-^2} \tag{43}$$

where $x_+^2 - x_-^2 < 0$ because $0 < x_+ < -x_-$.

The resulting integral

$$I = \frac{\sqrt{2}}{\omega_b}\int_0^{\pi} \frac{d\theta}{\sqrt{1 + \xi\cos\theta}} \tag{44}$$

gives rise to the series

$$I = \frac{\sqrt{2}\pi}{\omega_b}\sum_{j=0}^{\infty}(-1)^j \begin{pmatrix} -1/2 \\ j \end{pmatrix} \begin{pmatrix} -1/2 \\ 2j \end{pmatrix} \xi^{2j}. \tag{45}$$

which is similar to the one derived above for the Duffing oscillator and converges for all $|\xi| < 1$.

When $\mathcal{E} = U(x_M)$ then $x_3 = x_-$ (remember that $\lambda > 0$) and $\xi = 1$. We appreciate that the series (45) converges for all values of the energy for which there is periodic motion.

Equation (44) gives us a simple and exact expression for the period of the anharmonic oscillator (36) that requires numerical integration to obtain results for a given set of potential parameters. On the other hand, equation (45) provides approximate analytical expressions that one makes as accurate as desired by simply adding a sufficiently large number of terms. The choice of one or another depends on the particular application.

Following a different procedure Apostol[2] derived the exact expression for the period of the quadratic–cubic oscillator

$$T = \sqrt{\frac{3}{2\lambda}} \frac{4}{\omega_0 \sqrt{x_+ - x_3}} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}},\ k^2 = \frac{x_+ - x_-}{x_+ - x_3}. \tag{46}$$

The expansion of this equation in powers of k^2 is also convergent for all values of the energy consistent with periodic motion because $k^2 < 1$.

VI. CONCLUSIONS

In this paper we present a straightforward systematic procedure for constructing exact and approximate expressions for the period of anharmonic oscillators. The recipe is simple: first, we factor the function $Q(x)$ and obtain the turning points and the function $R(x)$ as in equation (10). Second, we obtain the maximum and minimum values of $R(x)$ in the interval between the turning points which determine the optimum value of ω. Thus we are left with an exact expression for the period that we may use in numerical applications. In addition to it, we may expand this exact expression in a Taylor series in order to obtain partial sums that become analytical expressions for the period of increasing accuracy. These partial sums converge to the exact result for all values of the energy that give rise to periodic motion.

The method proposed in this paper is not restricted to the period of anharmonic oscillators with polynomial potentials. We may, for example, expand a given arbitrary potential $U(x)$ about its minimum to any desired degree and then apply the approach developed above. Moreover, some other problems have been expressed in terms of integrals of the form (9), such as, for example, the deflection of light by a massive body or the precession of a planet orbiting around a star[3]. Recently, we have already applied a variant of present approach to such problems[4,5].

There is a wide range of interesting applications for present method and for that reason we believe that it is suitable for teaching in advanced undergraduate courses on classical mechanics.

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